ASU Sound & Vibration Lab

Faculty of Engineering
Ain Shams University

Acoustics

Sound & Vibration Measurements and Analysis of Simple Mechanical Systems

Dr. Tamer Elnady – Dr. Wael Akl – Dr. Adel Elsabbagh
aelsabbagh@gmail.com

#5: Analysis of Simple Mechanical Systems

Single Degree of Freedom

Damped-Forced SDOF

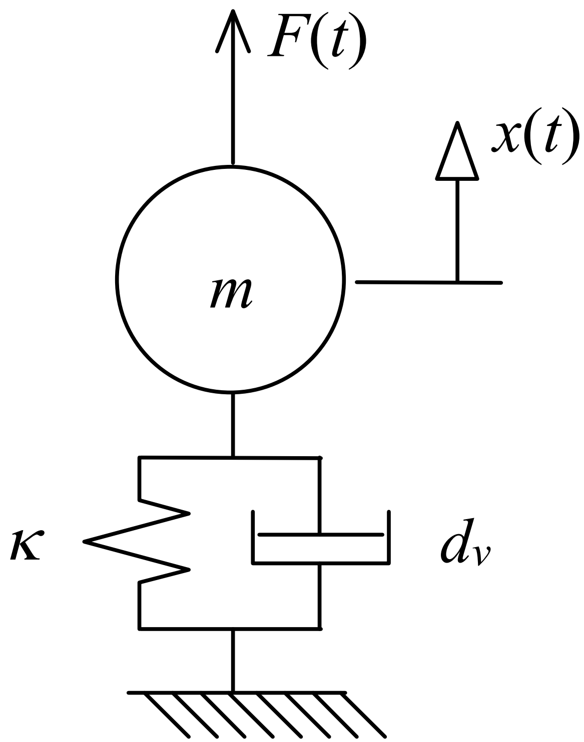
$$m\frac{d^2x(t)}{dt^2} = F_x\left(x(t), \frac{dx(t)}{dt}, t\right)$$

$$F_x = -\kappa x(t) - d_v\frac{dx(t)}{dt} + F(t)$$

$$m\frac{d^2x(t)}{dt^2} + d_v\frac{dx(t)}{dt} + \kappa x(t) = F(t)$$

$$\frac{d^2x(t)}{dt^2} + 2\delta\frac{dx(t)}{dt} + \omega_0^2 x(t) = g(t)$$

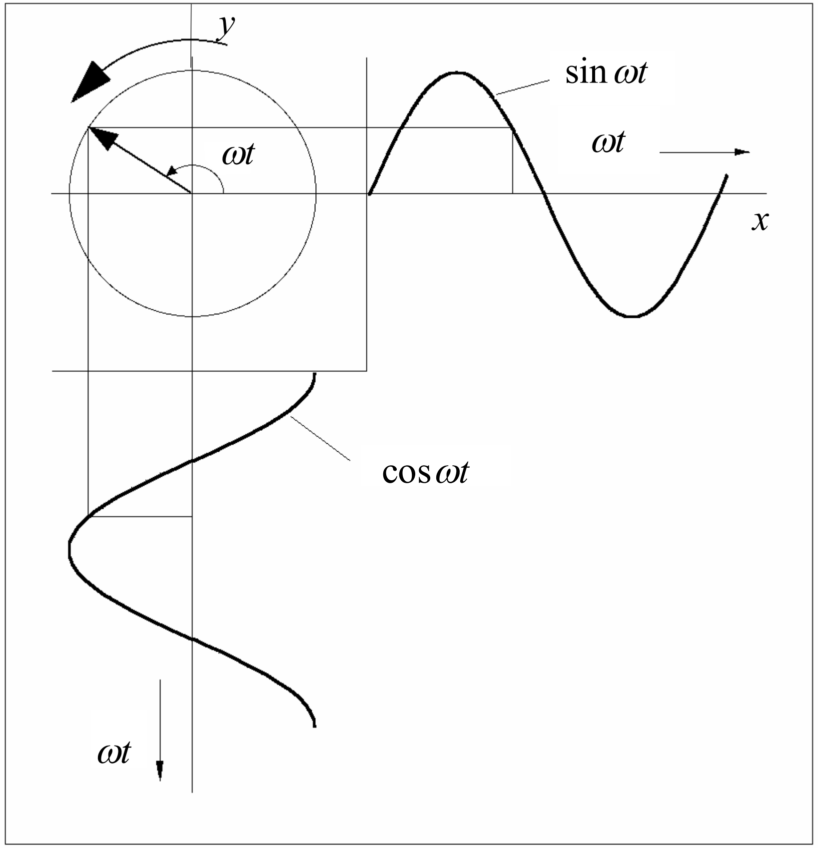
$$\omega_0 = \sqrt{\kappa/m} \qquad \delta = d_v/2m \qquad g(t) = F(t)/m$$



Complex Numbers and Rotating Vectors

Euler's identity

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$



Complex Numbers and Rotating Vectors

Example

Express the following harmonic function in terms of complex vectors

$$F(t) = \hat{F}\sin(\omega t)$$

Answer

$$F(t) = \mathrm{Im}\left(\mathbf{F}(t)\right) = \mathrm{Im}\left(\hat{F}e^{i\omega t}\right)$$
$$= \mathrm{Im}\left(\hat{F}\left(\cos(\omega t) + i\sin(\omega t)\right)\right)$$

or

$$F(t) = \mathrm{Re}\left(\mathbf{F}(t)\right) = \mathrm{Re}\left(\hat{F}e^{i(\omega t - \pi/2)}\right)$$

Damped-Forced SDOF

$$\mathbf{g}(t) = \hat{g}e^{i\omega t}$$

$$\mathbf{x}_p(t) = \hat{x}_p e^{i\varphi} e^{i\omega t} = \hat{\mathbf{x}}_p e^{i\omega t}$$

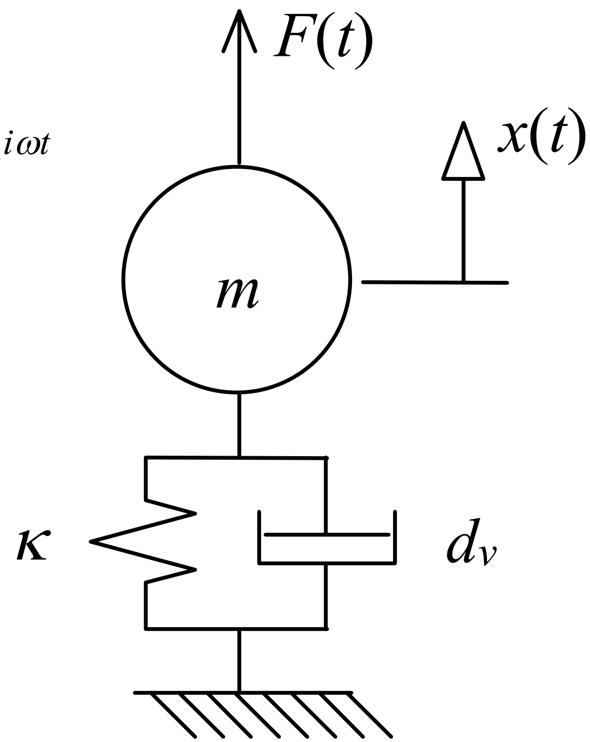
$$-\omega^2 \hat{\mathbf{x}}_p e^{i\omega t} + i2\omega\delta \hat{\mathbf{x}}_p e^{i\omega t} + \omega_0^2 \hat{\mathbf{x}}_p e^{i\omega t} = \hat{g}e^{i\omega t}$$

$$\hat{\mathbf{x}}_p = \frac{\hat{g}}{(\omega_0^2 - \omega^2) + i2\delta\omega}$$

$$\hat{\mathbf{x}}_p = \left|\hat{\mathbf{x}}_p\right| e^{i\varphi}$$

$$\left|\hat{\mathbf{x}}_p\right| = \frac{\hat{g}}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\delta\omega)^2}}$$

$$\varphi = \arctan\frac{2\delta\omega}{\omega^2 - \omega_0^2} + n\pi$$

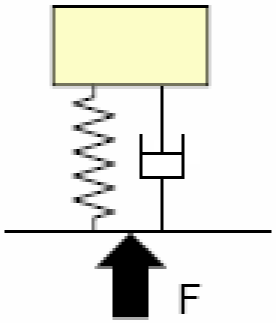


Multi Degree-of-Freedom

Response Models

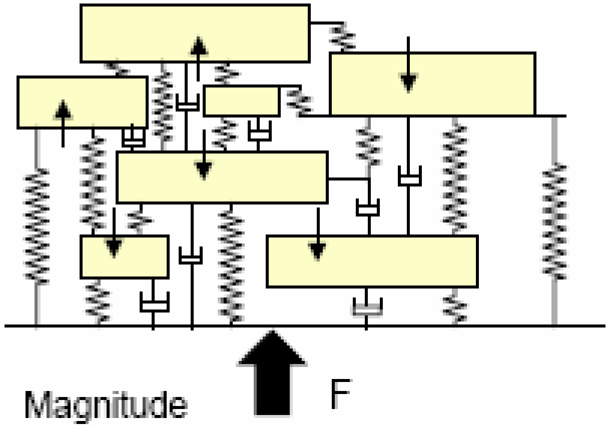
Single Degree of Freedom

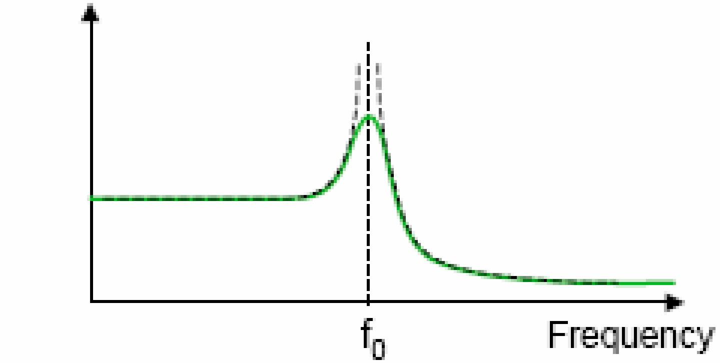
SDOF

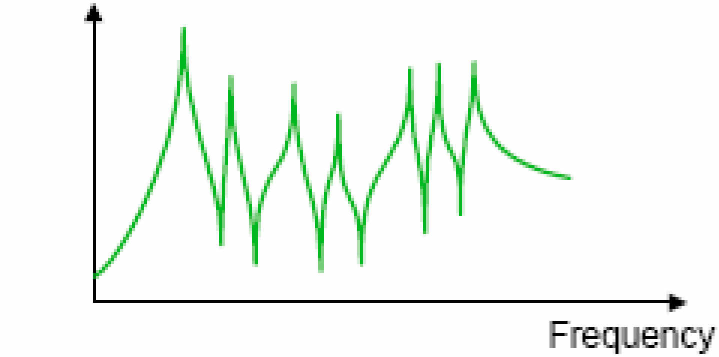


Multi Degree of Freedom

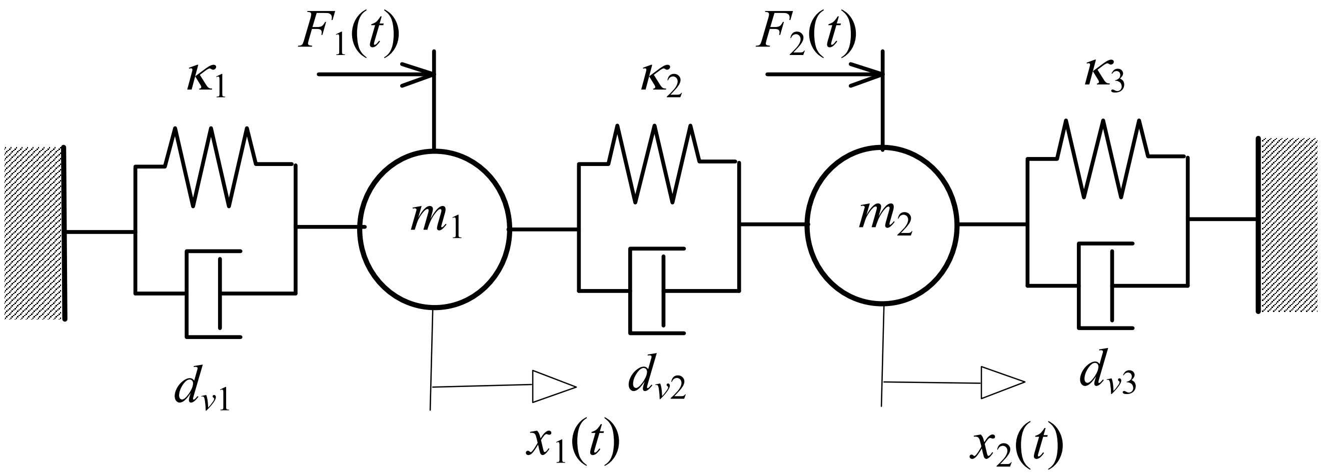
MDOF



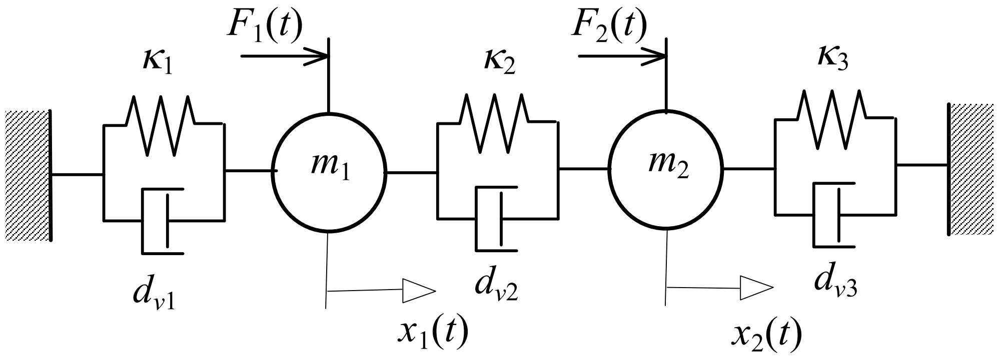




Two Degree-of-Freedom System



Two Degree-of-Freedom System



$$m_1 \frac{d^2 x_1(t)}{dt^2} + d_{v1} \frac{dx_1(t)}{dt} + d_{v2}\left(\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt} \right) +$$

$$+ \kappa_1 x_1(t) + \kappa_2 \left(x_1(t) - x_2(t) \right) = F_1(t)$$

$$m_2 \frac{d^2 x_2(t)}{dt^2} - d_{v2}\left(\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt} \right) + d_{v3} \frac{dx_2(t)}{dt} -$$

$$- \kappa_2 (x_1(t) - x_2(t)) + \kappa_3 x_2(t) = F_2(t)$$

Two Degree-of-Freedom System

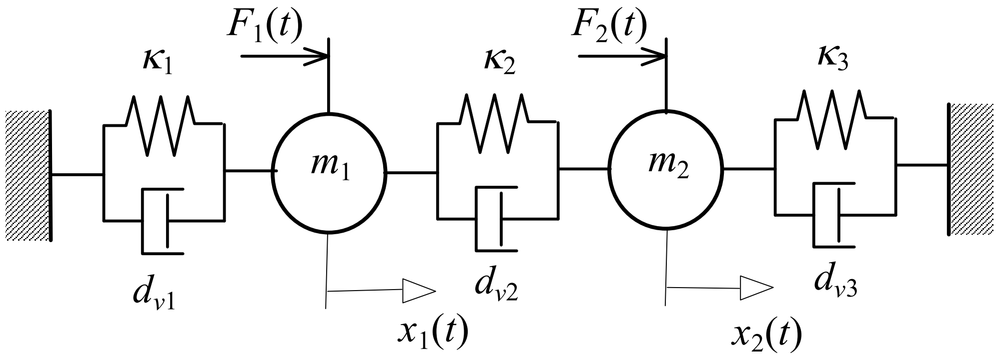
$$[\mathbf{M}]\cdot\frac{d^2\vec{x}}{dt^2}+[\mathbf{D}]\cdot\frac{d\vec{x}}{dt}+[\mathbf{K}]\cdot\vec{x}=\vec{F}$$

$$\vec{x}(t)=\begin{Bmatrix}x_1(t)\\x_2(t)\end{Bmatrix}\qquad \vec{F}(t)=\begin{Bmatrix}F_1(t)\\F_2(t)\end{Bmatrix}$$

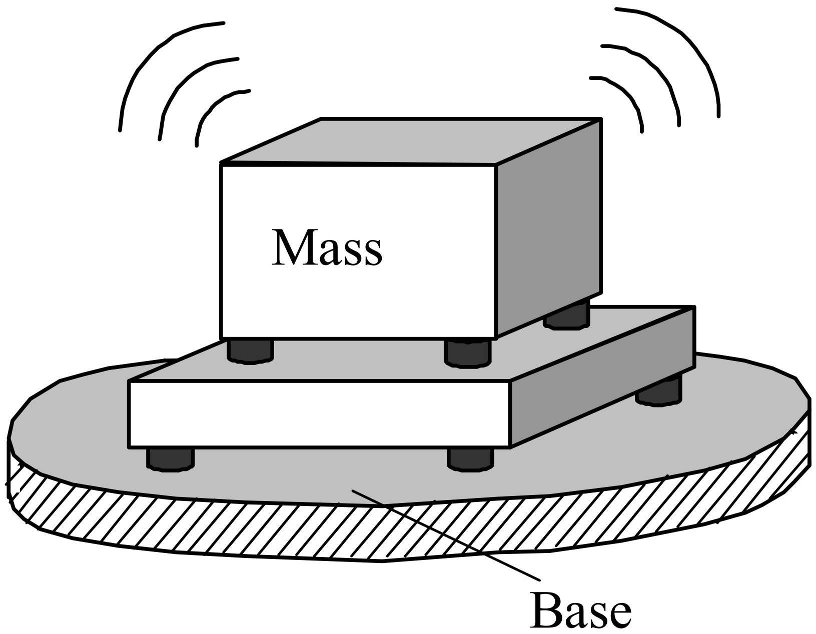
$$[\mathbf{M}]=\begin{bmatrix}m_1 & 0\\0 & m_2\end{bmatrix}$$

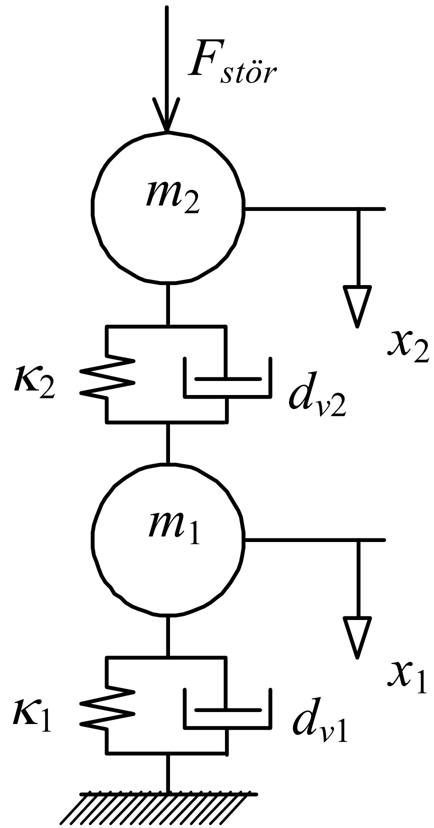
$$[\mathbf{D}]=\begin{bmatrix}d_{v1}+d_{v2} & -d_{v2}\\-d_{v2} & d_{v2}+d_{v3}\end{bmatrix}$$

$$[\mathbf{K}]=\begin{bmatrix}\kappa_1+\kappa_2 & -\kappa_2\\-\kappa_2 & \kappa_2+\kappa_3\end{bmatrix}$$

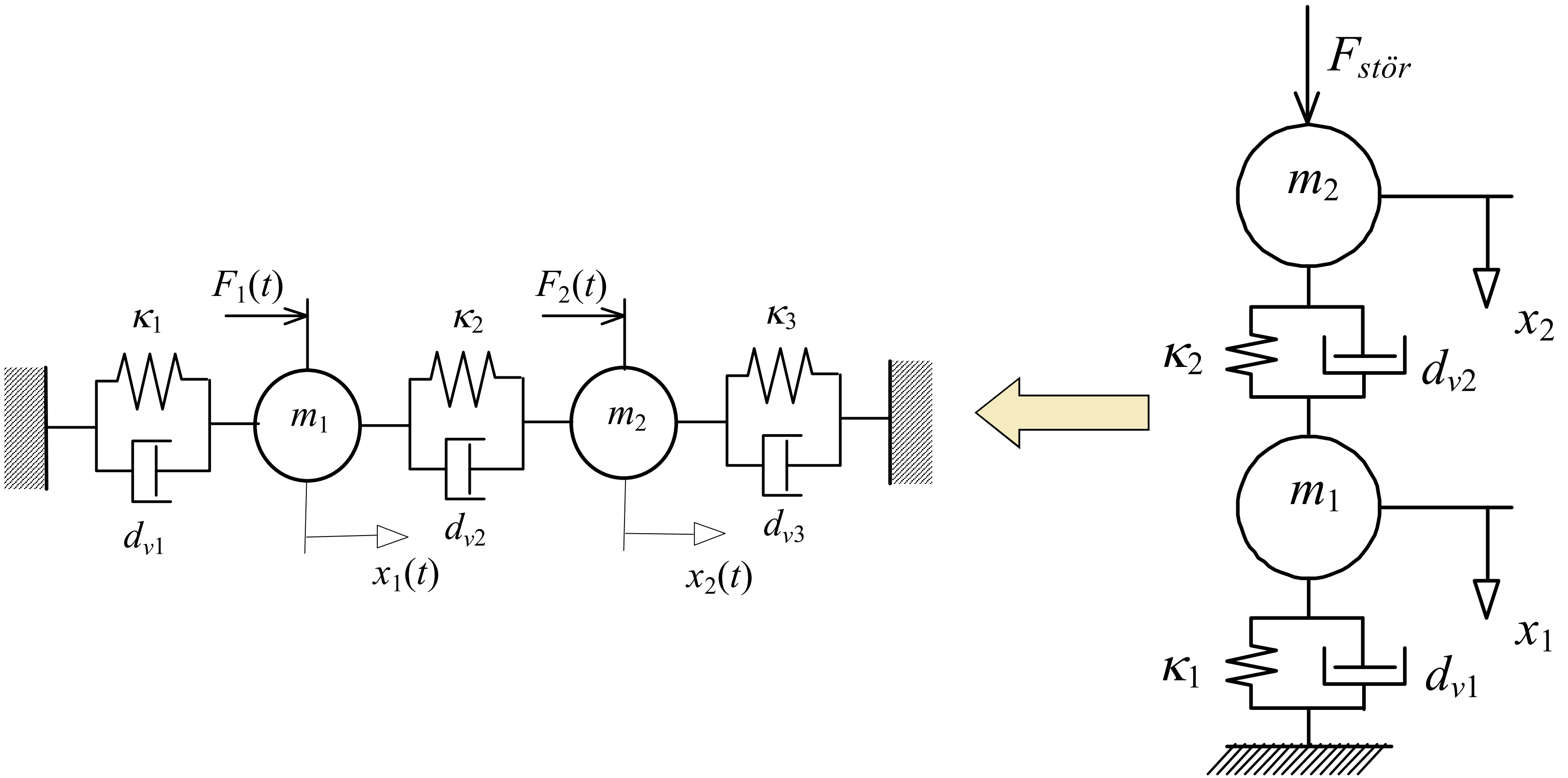


Double Elastic Mounting





Double Elastic Mounting



Double Elastic Mounting

$$\mathbf{F}_2(t) = \hat{\mathbf{F}}_2 e^{i\omega t} \qquad \mathbf{x}_{\mathbf{1p}}(t) = \hat{\mathbf{x}}_{\mathbf{1p}} e^{i\omega t} \qquad \mathbf{x}_{2\mathbf{p}}(t) = \hat{\mathbf{x}}_{2\mathbf{p}} e^{i\omega t}$$

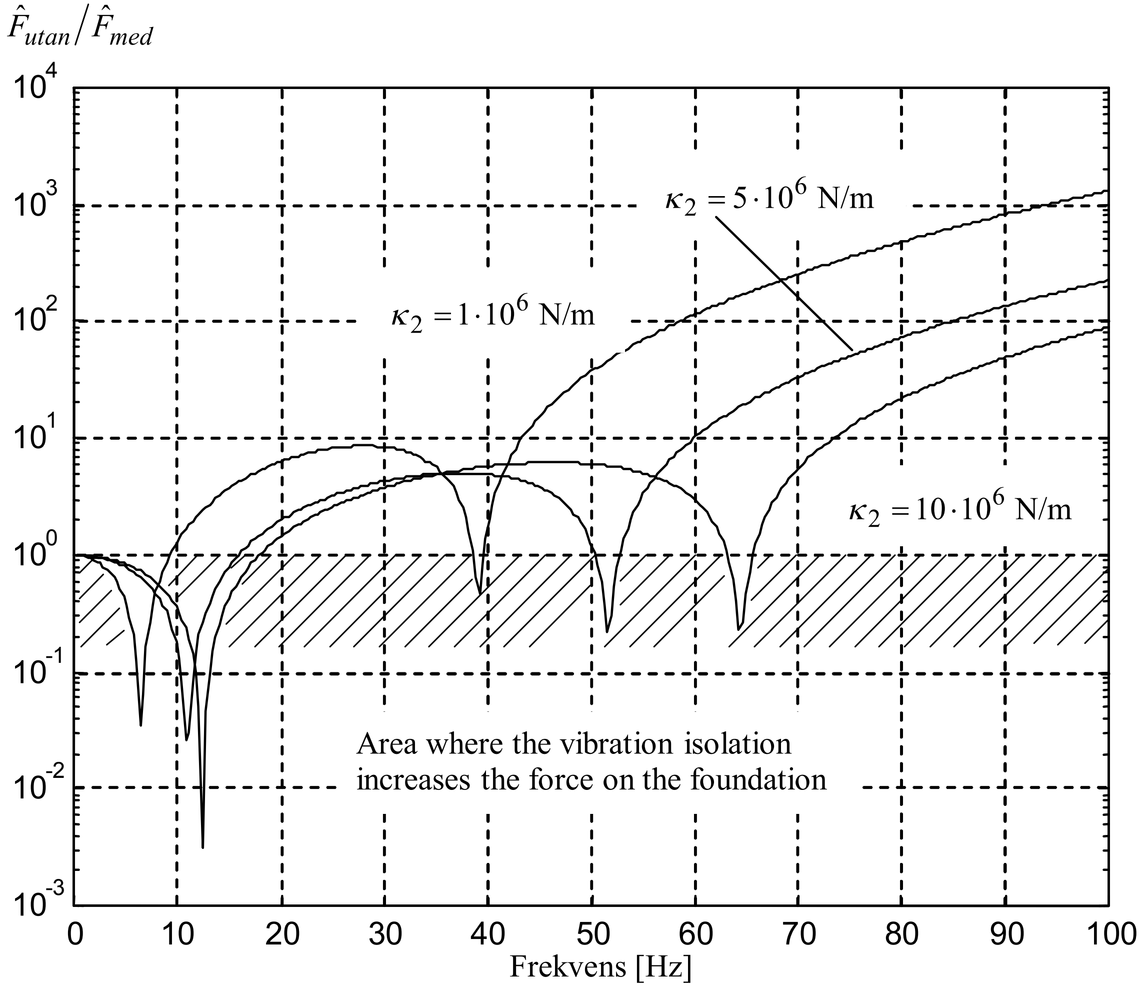
$$-\omega^2 [\mathbf{M}] \cdot \left\{ \hat{\mathbf{x}}_p \right\} + i\omega [\mathbf{D}] \cdot \left\{ \hat{\mathbf{x}}_p \right\} + [\mathbf{K}] \cdot \left\{ \hat{\mathbf{x}}_p \right\} = \left\{ \hat{\mathbf{F}} \right\}$$

Free Oscillations

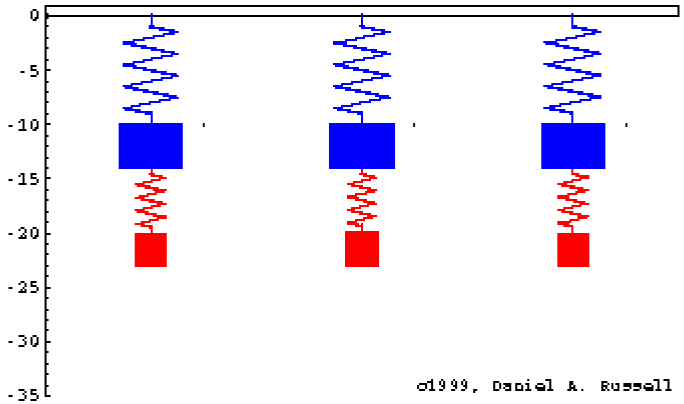
$$-\omega^2 [\mathbf{M}] \cdot \{\hat{\mathbf{x}}\} + [\mathbf{K}] \cdot \{\hat{\mathbf{x}}\} = \{0\}$$

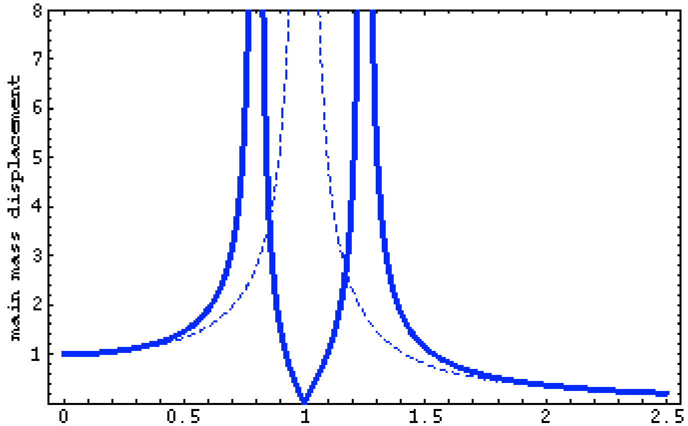
$$\det(-\omega^2 [\mathbf{M}] + [\mathbf{K}]) = 0$$

Double Elastic Mounting

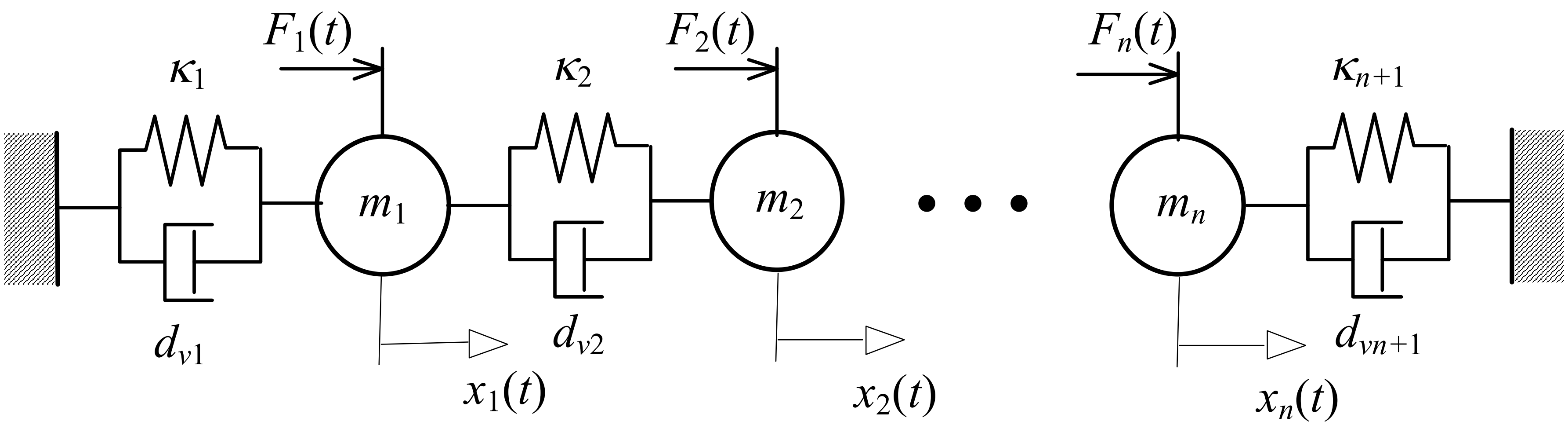


Vibration Absorber





Multi Degree-of-Freedom System



Multi Degree-of-Freedom System

$$[\mathbf{M}] = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & m_n \end{bmatrix}$$

$$[\mathbf{D}] = \begin{bmatrix} d_{v1} + d_{v2} & -d_{v2} & 0 & \cdot & \cdot & \cdot \\ -d_{v2} & d_{v2} + d_{v3} & -d_{v3} & 0 & \cdot & \cdot \\ 0 & -d_{v3} & \bullet & \bullet & \cdot & \cdot \\ \cdot & \cdot & \bullet & \bullet & \bullet & \cdot \\ \cdot & \cdot & 0 & -d_{vn-1} & d_{vn-1} + d_{vn} & -d_{vn} \\ \cdot & \cdot & \cdot & 0 & -d_{vn} & d_{vn} + d_{vn+1} \end{bmatrix}$$

Multi Degree-of-Freedom System

$$[\mathbf{K}] = \begin{bmatrix} \kappa_1 + \kappa_2 & -\kappa_2 & 0 & \cdot & \cdot & \cdot \\ -\kappa_2 & \kappa_2 + \kappa_3 & -\kappa_3 & 0 & \cdot & \cdot \\ 0 & -\kappa_3 & \bullet & \bullet & \cdot & \cdot \\ \cdot & \cdot & \bullet & \bullet & \bullet & \cdot \\ \cdot & \cdot & 0 & -\kappa_{n-1} & \kappa_{n-1} + \kappa_n & -\kappa_n \\ \cdot & \cdot & \cdot & 0 & -\kappa_n & \kappa_n + \kappa_{n+1} \end{bmatrix}$$

Frequency Response Functions

Quantity	Relation
Dynamic flexibility or *Receptance* $\mathbf{H}(\omega)$	$\mathbf{H}(\omega) = \mathbf{x}(\omega)/\mathbf{F}(\omega)$
Mobility or *mechanical admittance* $\mathbf{Y}(\omega)$	$\mathbf{Y}(\omega) = \mathbf{v}(\omega)/\mathbf{F}(\omega)$

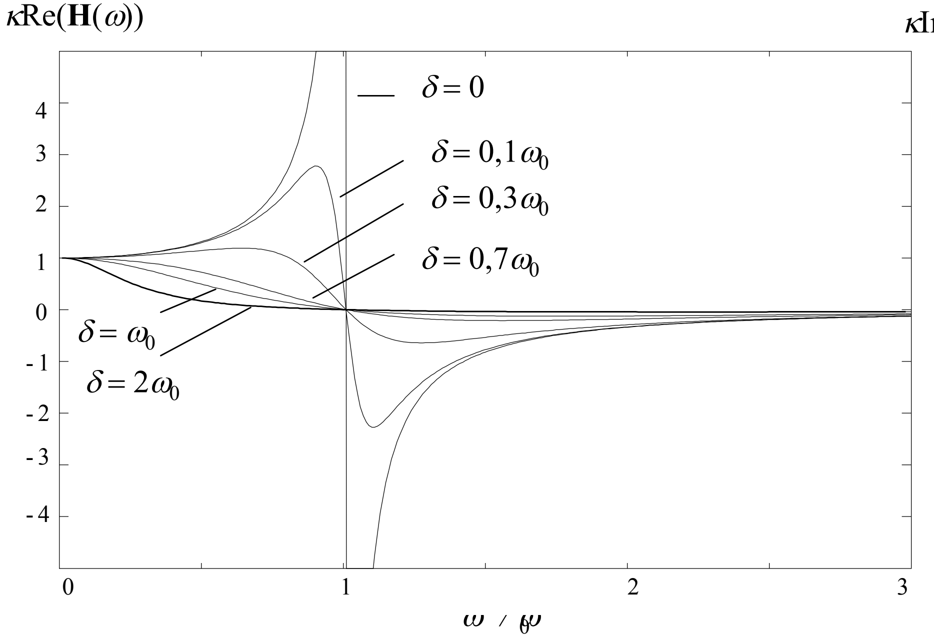
$$\mathbf{H}(\omega) = \frac{\mathbf{x}(\omega)}{\mathbf{F}(\omega)} = \frac{1}{-\omega^2 m + i\omega 2m\delta + \kappa} = \frac{1/\kappa}{1-(\omega/\omega_0)^2 + i2(\omega\delta/\omega_0^2)}$$

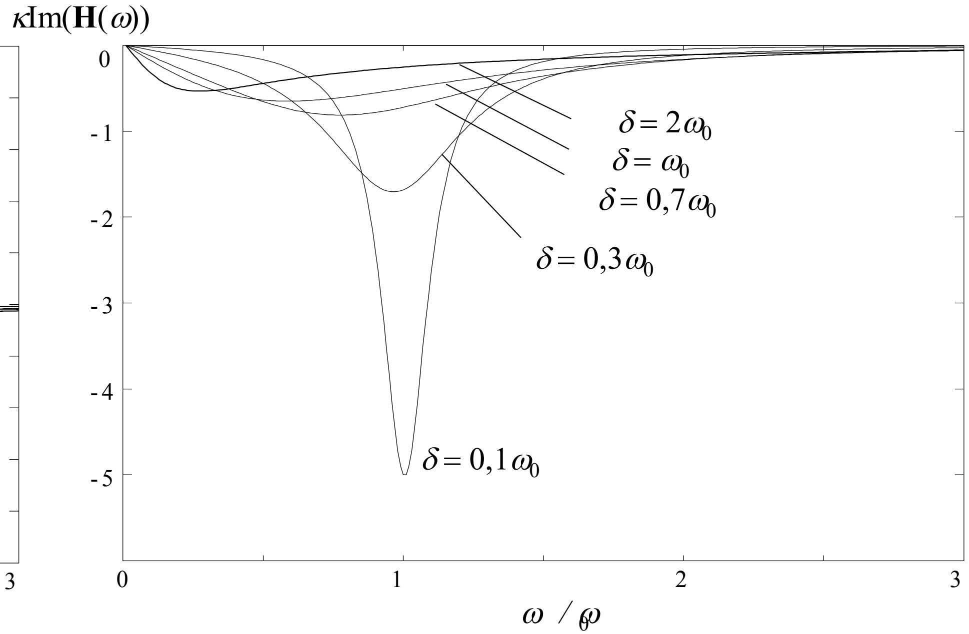
Frequency Response Functions

Quantity	Relation
Accelerance $\mathbf{A}(\omega)$	$\mathbf{A}(\omega) = \mathbf{a}(\omega)/\mathbf{F}(\omega)$
Dynamic stiffness $\boldsymbol{\kappa}(\omega)$	$\boldsymbol{\kappa}(\omega) = \mathbf{F}(\omega)/\mathbf{x}(\omega)$
Mechanical impedance $\mathbf{Z}(\omega)$	$\mathbf{Z}(\omega) = \mathbf{F}(\omega)/\mathbf{v}(\omega)$
Acoustic impedance $\mathbf{Z}(\omega)$	$\mathbf{Z}(\omega) = \mathbf{p}(\omega)/\mathbf{Q}(\omega)$
Specific impedance $\mathbf{Z}(\omega)$	$\mathbf{Z}(\omega) = \mathbf{p}(\omega)/\mathbf{u}(\omega)$

Frequency Response Functions

$$\mathbf{H}(\omega)=\frac{\mathbf{x}(\omega)}{\mathbf{F}(\omega)}=\frac{1}{-\omega^2 m+i\omega 2m\delta+\kappa}=\frac{1/\kappa}{1-(\omega/\omega_0)^2+i2(\omega\delta/\omega_0^2)}$$





Frequency Response Functions

$$\left|\mathbf{H}(\omega)\right| = \frac{1/\kappa}{\sqrt{(1-(\omega/\omega_0)^2)^2 + (2\,\omega\delta/\omega_0^2)^2}} \qquad \varphi(\omega) = \arctan\frac{2\,\omega\delta/\omega_0{}^2}{(\omega/\omega_0)^2 - 1}$$